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Characteristic Exponents and Diagonally Dominant Linear Differential Systems

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1. INTRODUCTION

Let $A(t) = (a_{ij}(t))$ be a complex $n \times n$ matrix valued function with elements continuous on the real line such that $A(t + T) = A(t)$ for some $T > 0$. We consider the differential equations

$$\frac{dx}{dt} = A(t)\, x, \tag{1.1}$$

$$\frac{dX}{dt} = A(t)\, X, \tag{1.2}$$

where in (1.1) x denotes an n-column vector and in (1.2) X denotes an $n \times n$ matrix. By a well known result, essentially due to Floquet [3] and dating back to 1883, if $X(t)$ is the solution of (1.2) satisfying the initial condition $X(0) = I$, the identity matrix (or any nonsingular solution), then $X(t)$ may be represented in the form

$$X(t) = Q(t)\, e^{Bt} \tag{1.3}$$

where B is a constant matrix and $Q(t + T) = Q(t)$. Qualitatively the asymptotic behavior of solutions of (1.1) is determined entirely by the real parts of the eigenvalues of B provided B has at most one eigenvalue with zero real part. (See for example [1, Chap. 2] or [5, Chap. 3]). It is therefore a fundamental problem in the theory of linear periodic differential equations to determine the real parts of the eigenvalues of the matrix B given $A(t)$. Our first main result is a modest contribution to this problem. Our motivation comes from a beautiful and elementary result due to S. A. Gerschgorin [4]

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which states that if $A = (a_{ij})$ is an $n \times n$ complex constant matrix then every eigenvalue of A is contained in one of the closed discs

$$| z - a_{ii} | \leqslant \sum_{\substack{j=1 \\ j \neq i}}^{n} | a_{ij} | \qquad i = 1,..., n.$$

In fact if the coefficient matrix $A(t)$ in (1.1) is constant then the first part of our result is implied by Gerschgorin's result (see Theorem 1). The proof of the first part of our first result is motivated by the proof of Gerschgorin's theorem as given in [6].

Our second main result is an extention of a corollary of our first main result to a class of linear differential systems which are not necessarily periodic or even bounded. As an application of our second result we obtain a result concerning the geometry of trajectories of the linear differential system

$$\frac{dx}{dt} = [B + C(t)]\, x$$

where B is a constant matrix with no purely imaginary eigenvalues and the moduli of the elements of $C(t)$ satisfy certain bounds depending on B.

2. Bounds for the Real Parts of Characteristic Exponents

We recall briefly some terminology and known results concerning linear periodic differential systems. Proofs can be found in [1] and [5].

Let $X(t)$ be the solution of (1.2) satisfying $X(0) = I$. An eigenvalue of a matrix B such that (1.3) holds for some T-periodic matrix $Q(t)$ is called a *characteristic exponent* of the system (1.1). If $\Lambda = \{\lambda_1 ,..., \lambda_n\}$ denotes the set of eigenvalues of such a matrix B, each eigenvalue repeated as often as its multiplicity, then Λ will be referred to as *a system of characteristic exponents* of (1.2). It is known that λ is a characteristic exponent of (1.1) if and only if there exists a nontrivial solution of (1.1) of the form

$$x(t) = p(t)\, e^{\lambda t} \qquad \text{where} \qquad p(t) = p(t + T)$$

and hence

$$\frac{dp}{dt} = - \lambda p + A(t)\, p. \tag{2.1}$$

If $X(t)$ is as above then the eigenvalues of $X(T) = e^{BT}$ are called the *multipliers* of the system (1.1). Since there is only one solution of (1.2) satisfying $X(0) = I$ the multipliers of (1.1) are uniquely determined by $A(t)$.

Moreover the eigenvalues of e^{BT} are the numbers $e^{\lambda T}$ where λ ranges over the eigenvalues of B so the real parts of the members of a system of characteristic exponents of (1.1) are uniquely determined by $A(t)$. However, in general, the imaginary parts are only determined within an integral multiple of $2\pi i/T$.

THEOREM 1. *Let $A(t)$ be as above. If for each $i = 1,\ldots, n$ we define*

$$r_i(t) = \sum_{\substack{j=1\\ j\neq i}}^{n} | a_{ij}(t)| , \tag{2.2}$$

$$c_i = \min_{[0,T]}(\mathrm{Re}(a_{ii}(t)) - r_i(t)) \tag{2.3}$$

$$d_i = \max_{[0,T]}(\mathrm{Re}(a_{ii}(t)) + r_i(t)) \tag{2.4}$$

then:

(a) *Every characteristic exponent of the system* (1.1) *is contained in one of the closed strips*

$$S_i : c_i \leqslant \mathrm{Re}(z) \leqslant d_i , \qquad i = 1,\ldots, n, \tag{2.5}$$

(b) *If J is a maximal connected subset of $\bigcup_{i=1}^{n} S_i$ and there are exactly ℓ distinct integers i such that $S_i \subseteq J$, then for any system Λ of characteristic exponents of* (1.1) *exactly ℓ elements of Λ counting multiplicities, are contained in J.*

Proof. To prove part (a) assume λ is a characteristic exponent of (1.1). there exists a complex column vector function

$$p(t) = \mathrm{col}(p_1(t),\ldots, p_n(t))$$

defined on $(-\infty, \infty)$ such that $p(t) \equiv p(t + T)$, the coordinates of p never vanish simultaneously and p is a solution of (2.1).

Let

$$M = \max_{1\leqslant i\leqslant n} \max_{t\in[0,T]} | p_i(t)| . \tag{2.6}$$

If k is an integer and $s \in [0, T]$ a number such that

$$| p_k(s)| = M, \tag{2.7}$$

then by periodicity the continuously differentiable function $|p_k|^2$ has a relative maximum at s and so according to (2.1)

$$\begin{aligned}
0 &= \frac{d}{dt}|p_k|^2(s) = p_k(s)\frac{d}{dt}\bar{p}_k(s) + \bar{p}_k(s)\frac{d}{dt}p_k(s) \\
&= p_k(s)\overline{\frac{d}{dt}p_k(s)} + \overline{p_k(s)}\frac{d}{dt}p_k(s) = 2\operatorname{Re}\left(\overline{p_k(s)}\frac{d}{dt}p_k(s)\right) \\
&= 2\operatorname{Re}\left(\overline{p_k(s)}\left(-\lambda p_k(s) + \sum_{j=1}^{n} a_{kj}(s)\, p_j(s)\right)\right) \\
&= 2(\operatorname{Re}(a_{kk}(s)) - \operatorname{Re}(\lambda))\,|p_k(s)|^2 + 2\operatorname{Re}\sum_{\substack{j=1\\ j\neq k}}^{n} a_{kj}(s)\, p_j(s)\,\overline{p_k(s)}.
\end{aligned}$$

Thus, by (2.6) and (2.7)

$$\begin{aligned}
|\operatorname{Re}(\lambda) - \operatorname{Re}(a_{kk}(s))|\, M^2 &= \left|\operatorname{Re}\sum_{\substack{j=1\\ j\neq k}}^{n} a_{kj}(s)\, p_j(s)\,\overline{p_k(s)}\right| \\
&\leqslant \sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}(s)|\,|p_j(s)|\,|\overline{p_k(s)}| \leqslant M^2\sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}(s)|
\end{aligned}$$

and since p is a nontrivial solution of (2.1) we have by (2.2)

$$|\operatorname{Re}(\lambda) - \operatorname{Re}(a_{kk}(s))| \leqslant \sum_{\substack{j=1\\ j\neq k}}^{n} |a_{kj}(s)| = r_k(s).$$

Using (2.3) and (2.4) we obtain

$$\operatorname{Re}(\lambda) \in [\operatorname{Re}(a_{kk}(s)) - r_k(s), \operatorname{Re}(a_{kk}(s)) + r_k(s)] \subseteq [c_k, d_k]$$

so $\lambda \in S_k$. This completes the proof of part (a) of Theorem 1.

To prove part (*b*) let J be a component (maximal connected subset) of $\bigcup_{i=1}^{n} S_i$. Clearly J is a strip

$$C \leqslant \operatorname{Re} z \leqslant D$$

where C is the minimum of the numbers c_i and D is the maximum of the numbers d_i such that $S_i \subseteq J$.

If for $i = 1, \ldots, n$ we define a closed ring

$$R_i : \exp c_i T \leqslant |z| \leqslant \exp d_i T \tag{2.8}$$

then

$$H : \exp CT \leqslant |z| \leqslant \exp DT \tag{2.9}$$

is a component of $\bigcup_{i=1}^{n} R_i$. Now a characteristic exponent λ of (1.1) is contained in the strip S_i if and only if the corresponding multiplier $e^{\lambda T}$ is contained in the corresponding ring R_i. The assertion of part (b) of Theorem 1 is therefore equivalent to the assertion:

If there are exactly ℓ integers i such that $R_i \subseteq H$ then exactly ℓ multipliers of (1.1), *counting multiplicities, are contained in H.*

For each number $u \in [0, 1]$ let $A(t, u) = (a_{ij}(t, u))$ be the $n \times n$ complex matrix function defined by

$$a_{ij}(t, u) = \begin{cases} u a_{ij}(t), & i \neq j, \\ a_{ii}(t), & i = j, \end{cases} \tag{2.10}$$

for i, $j = 1,\ldots, n$ and let $X(t, u)$ denote the $n \times n$ matrix such that

$$\frac{dX}{dt} = A(t, u)\, X, \qquad X(0, u) = I. \tag{2.11}$$

For $i = 1,\ldots, n$ and $u \in [0, 1]$ let the numbers $c_i(u)$ and $d_i(u)$ be defined for the matrix $A(t, u)$ in the same way the numbers c_i and d_i are defined for $A(t)$. From (2.2)–(2.4) and (2.10) it follows that

$$c_i(u) = \min_{[0,T]}[\mathrm{Re}(a_{ii}(t)) - u r_i(t)] \tag{2.12}$$

$$d_i(u) = \max_{[0,T]}[\mathrm{Re}(a_{ii}(t)) + u r_i(t)] \tag{2.13}$$

for $u \in [0, 1]$ and $i = 1,\ldots, n$. If for i and u in the same range we define closed rings

$$R_i(u) : \exp c_i(u)\, T \leqslant |z| \leqslant \exp d_i(u)\, T \tag{2.14}$$

then by (2.3), (2.4) and the above

$$R_i(u_1) \subseteq R_i(u_2) \qquad \text{if} \quad 0 \leqslant u_1 \leqslant u_2 \leqslant 1, \tag{2.15}$$

and $R_i(1) = R_i$. Consequently if for each $u \in [0, 1]$ we let $M(u)$ denote the number of distinct intergers i such that $R_i(u) \subseteq H$ $M(u)$ is constant so

$$M(1) = M(0). \tag{2.16}$$

Since the ring H is a component of $\bigcup_{i=1}^{n} R_i$ it follows by (2.9) that we may choose $\delta > 0$ so small that H is contained in the annular region between the two circles

$$\Gamma_1 : |z| = \exp CT - \delta$$
$$\Gamma_2 : |z| = \exp DT + \delta$$

and such that Γ_1 and Γ_2 do not intersect $\bigcup_{i=1}^n R_i$. Now by part (a) of Theorem 1 applied to $A(t, u)$ for $u \in [0, 1]$ every multiplier of the linear periodic system

$$\frac{dx}{dt} = A(t, u)\, x \tag{2.17}$$

must be contained in one of the closed rings $R_i(u)$, $i = 1,..., n$. Therefore by (2.15) and the above no multiplier of the system (2.17) is on Γ_1 or Γ_2 for all $u \in [0, 1]$. Now since the multipliers of (2.17) are the eigenvalues of $X(T, u)$ it follows from the argument principle that if for each $u \in [0, 1]$ $N(u)$ denotes the number of multipliers of (2.17) contained in H then

$$N(u) = \frac{1}{2\pi i} \oint_{\Gamma_2} \left(\frac{\partial}{\partial z} \frac{p(z, u)}{p(z, u)} \right) dz - \frac{1}{2\pi i} \oint_{\Gamma_2} \left(\frac{\partial}{\partial z} \frac{p(z, u)}{p(z, u)} \right) dz \tag{2.18}$$

where $p(z, u)$ is the polynomial in z defined by $p(z, u) = \det[X(T, u) - zI]$.

From (2.10) $A(t, u)$ depends continuously on u so by (2.11) and a standard theorem in differential equations the elements of $X(T, u)$ depend continuously on u. Hence the coefficients of the polynomial $p(z, u)$ depend continuously on u so from (2.18) we infer that the integer $N(u)$ is defined and a continuous function of u for $u \in [0, 1]$. Hence

$$N(0) = N(1). \tag{2.19}$$

Recalling the definition of $M(u)$ and observing that $A(t, 1) = A(t)$ part (b) of Theorem 1 is equivalent to the equality $M(1) = N(1)$ and by (2.16) and (2.19) this is equivalent to be equality

$$N(0) = M(0). \tag{2.20}$$

To prove (2.20) consider the matrix $X(T, 0)$. From (2.10) and (2.11) we see that

$$(T, 0) = \operatorname{diag}(\sigma_1 ,..., \sigma_n)$$

where

$$\sigma_i = \exp \int_0^T a_{ii}(t)\, dt \qquad i = 1,..., n,$$

so these numbers are the multipliers of the system (2.17) for $u = 0$. By (2.12) and (2.13)

$$| \sigma_i | \geqslant \exp(\min_{[0,T]} \operatorname{Re}(a_{ii}(t)\; T)) = \exp c_i(0)\; T,$$

and

$$\sigma_i \mid \leqslant \exp(\max_{[0,T]} \operatorname{Re}(a_{ii}(t)\, T)) = \exp d_i(0)\, T$$

so by (2.14)

$$\sigma_i \in R_i(0) \qquad i = 1,..., n.$$

Hence we see at once that $R_i(0) \subseteq H$ implies $\sigma_i \in H$. Conversely, if $\sigma_i \in H$ $R_i(0)$ must intersect H so by (2.15) $R_i = R_i(1)$ intersects H and since H is a component of $\bigcup_{i=1}^n R_i$, $R_i(0) \subseteq R_i \subseteq H$. This proves the equality (2.20) and by an earlier remark the proof of part (b) of Theorem 1 is complete.

The folowing consequence of Theorem 1 motivates the result of the next section.

COROLLARY. *If*

$$| \operatorname{Re}(a_{ii}(t))| > r_i(t) \tag{2.21}$$

for $i = 1,..., n$ *and* $t \in (-\infty, \infty)$ *then there are no purely imaginary characteristic exponents of* (1.1) *and if there are exactly k integers i such that* $\operatorname{Re} a_{ii}(t) > 0$ *for all t then for every system* Λ *of characteristic exponents of* (1.1) *exactly k elements of* Λ *have positive real part.*

Proof. From (2.2) to (2.5) it follows that if (2.21) holds for all t then the strip S_i cannot intersect the imaginary axis. Moreover if (2.21) holds for all t and $1 \leqslant i \leqslant n$ then the number of distinct integers i such that S_i is contained in the half plane $\operatorname{Re} z > 0$ is precisely the number of distinct integers i such that $\operatorname{Re} a_{ii}(t) > 0$ for all $t \in (-\infty, \infty)$. Thus the assertion of the corollary follows from part (b) of Theorem 1.

3. Diagonally Dominant Linear Differential Systems

In this section we let $A(t) = (a_{ij}(t))$ denote an $n \times n$ complex matrix function with entries which are continuous on the real line. We assume the existence of a number $\delta > 0$ such that

$$| \operatorname{Re}(a_{ii}(t))| \geqslant \sum_{\substack{j=1\\ j\neq i}}^{n} | a_{ij}(t)| + \delta \tag{3.1}$$

for all $t \in (-\infty, \infty)$ and $i = 1,..., n$. We note that if $A(t)$ is periodic and we assume only the weaker inequalities (2.21) then there will exist a number $\delta > 0$ such that the inequalities (3.1) hold. However, in this section *the entries of* $A(t)$ *are not even assumed to be bounded on* $(-\infty, \infty)$.

In this section we let K^n denote the set of n-dimensional complex column vectors considered as an n-dimensional complex vector space. Lower case Greek letters will be used exclusively for real or complex constants. If

$$c = \mathrm{col}(\gamma_1, \ldots, \gamma_n) \in K^n$$

we set

$$\| c \| = \max_{1 \leqslant i \leqslant n} | \gamma_i | . \tag{3.2}$$

If c is as above c^* will denote the row vector given by

$$c^* = (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$$

and $| c | = \sqrt{c^* c}$ the usual Euclidean norm of $c \in K^n$.

The following theorem shows that the equilibrium solution $x = 0$ of

$$\frac{dx}{dt} = A(t)\, x \tag{3.3}$$

possesses a sort of "*nonautonomous saddle point property*" *under conditions* (3.1) (see [5 Chap. 3]). Using the terminology of [2] the theorem proves the existence of an *exponential dichotomy* of (3.3) under the conditions (3.1).

THEOREM 2. *Let* (3.1) *hold for all* $t \in (-\infty, 1\infty)$ *and* $i = 1, \ldots, n$. *If there are exactly k integers i such that* $\mathrm{Re}(a_{ii}(t)) > 0$ *for all* $t \in (-\infty, \infty)$ *then*:

(a) *There exist k independent solutions* $y^1, \ldots, y^k$ *of the linear differential system* (3.3) *such that if* $x(t)$ *is a nontrivial solution of* (3.3) *of the form*

$$x(t) = \alpha_1 y^1(t) + \cdots + \alpha_k y^k(t)$$

then $\| x(t) \|$ *is strictly increasing on* $(-\infty, \infty)$, $\| x(t) \| \to \infty$ *exponentially as* $t \to \infty$, *and* $\| x(t) \| \to 0$ *exponentially as* $t \to \infty$. *In fact if* $\delta > 0$ *is as in* (3.1) *then*

$$\| x(t_1) \| \exp \delta(t_2 - t_1) \leqslant \| x(t_2) \| \qquad \text{if} \quad t_1 \leqslant t_2 .$$

(b) *There exist* $n - k$ *independent solutions* $z^1, \ldots, z^{n-k}$ *of* (3.3) *such that if* $x(t)$ *is a nontrivial solution of* (3.3) *of the form*

$$x(t) = \alpha_1 z^1 + \cdots + \alpha_{n-k} z^{n-k}$$

then $\| x(t) \|$ *is strictly decreasing on* $(-\infty, \infty)$, $\| x(t) \| \to 0$ *exponentially as* $t \to \infty$, *and* $\| x(t) \| \to \infty$ *exponentially as* $t \to -\infty$. *In fact if* $\delta > 0$ *is as in* (3.1) *then*

$$\| x(t_2) \| \leqslant \| x(t_1) \| \exp - \delta(t_2 - t_1) \qquad \text{if} \quad t_1 \leqslant t_2 .$$

(c) *The solutions* $y^1, \ldots, y^k$, $z^1, \ldots, z^{n-k}$ *form a basis of the solution space of* (3.3) *and the equilibrium solution* $x = 0$ *is the only solution of* (3.3) *bounded on* $(-\infty, \infty)$.

Before turning to the proof of Theorem 2 we mention one simple application. Let B be an $n \times n$ complex matrix which has no purely imaginary eigenvalues. It is well known that there exists a number $r > 0$ such that if $C(t) = (c_{ij}(t))$ is a continuous $n \times n$ complex matrix such that $|c_{ij}(t)| \leq r$ for $t \in (-\infty, \infty)$, $i, j = 1, \ldots, n$ then the statements (a), (b) and (c) of Theorem 2 (except those concerning monotonicity) hold for the linear differential system

$$\frac{dx}{dt} = [B + C(t)]\, x$$

if k is the number of eigenvalues of B with positive real part.

This result can also be established as a consequence of Theorem 2. In fact given $\epsilon > 0$ there exists a matrix T such that if

$$J = T^{-1}BT$$

the diagonal elements of J are the eigenvalues of B, every element on the super diagonal of J is either ϵ or zero, and every element of J not on the diagonal or super diagonal of J is zero. The change of variables $x = Ty$ transforms the above differential system into

$$\frac{dy}{dt} = A(t)\, y, \qquad A(t) = J + T^{-1}C(t)\, T.$$

Consequently, given $\epsilon > 0$ and T, bounds for the elements of $C(t)$ can be chosen so that $A(t)$ will satisfy the hypothesis of Theorem 2.

The use of Theorem 2 to establish this result seems novel, since every other proof known to the author uses either successive approximations or fixed point arguments applied to a system of integral equations whereas the proof of Theorem 2 uses only simple differential inequalities and elementary compactness arguments.

The proof of Theorem 2 uses five preliminary lemmas.

LEMMA 2.1. *Under the assumptions of Theorem* 2 *if* $x(t)$ *is a nontrivial solution of* (3.3) *then* $\|x(t)\|$ *cannot have a relative maximum anywhere on* $(-\infty, \infty)$.

Proof. Assume on the contrary that $x(t)$ is a nontrivial solution of (3.3) and $\|x(t)\|$ has a relative maximum at $t = s$. If $x(t) = \mathrm{col}(x_1(t), \ldots, x_n(t))$

and $\| x(s)\| = | x_h(s)|$, $| x_h(s)| \neq 0$ and by (3.2) $| x_h(t)|^2$ has a ralative maximum at $t = s$. Hence by (3.3),

$$
\begin{aligned}
0 = \frac{1}{2}\left| \frac{d}{dt} | x_h |^2 (s) \right| &= \left| \operatorname{Re} \left(\overline{x_h(s)} \frac{dx_h}{dt}(s) \right) \right| \\
&= \left| \operatorname{Re}(a_{hh}(s) \, | x_h(s)|^2) + \operatorname{Re} \left(\sum_{\substack{j=1 \\ j \neq h}}^{n} a_{hj}(s) \, x_j(s) \, \overline{x_h(s)} \right) \right| \\
&\geqslant | \operatorname{Re}(a_{hh}(s))| \, | x_h(s)|^2 - \sum_{\substack{j=1 \\ j \neq h}}^{n} | a_{hj}(s)| \, | x_j(s)| \, | \overline{x_h(s)}| \\
&\geqslant \left[| \operatorname{Re}(a_{hh}(s))| - \sum_{\substack{j=1 \\ j \neq h}}^{n} | a_{hj}(s)| \right] | x_h(s)|^2,
\end{aligned}
$$

and so

$$
| \operatorname{Re}(a_{hh}(s))| \leqslant \sum_{\substack{j=1 \\ j \neq h}}^{n} | a_{hj}(s)|
$$

which contradicts (3.1) for $i = h$. This contradiction proves the lemma.

LEMMA 2.2. *Under the assumptions of Theorem* 1 *if* $x(t)$ *is a nontrivial solution of* (3.3) *such that for some* $s \in (-\infty, \infty)$ $\| x(s)\| = | x_h(s)|$ *and* $\operatorname{Re}(a_{hh}(t)) > 0$ *for all* $t \in (-\infty, \infty)$ *then* $\| x(t)\|$ *is strictly increasing on the interval* $[s, \infty)$.

Proof. Since $x(t)$ is a nontrivial solution of (3.3) $| x_h(s)| > 0$. Therefore using (3.2) and (3.3) we have

$$
\begin{aligned}
\frac{1}{2} \frac{d}{dt} | x_h |^2 (s) &= \operatorname{Re} | a_{hh}(s)| \, | x_h(s)|^2 + \operatorname{Re} \left(\sum_{\substack{j=1 \\ j \neq h}}^{n} a_{hj}(s) \, x_j(s) \, \overline{x_h(s)} \right) \\
&\geqslant \operatorname{Re}(a_{hh}(s)) \, | x_h(s)|^2 - \sum_{\substack{j=1 \\ j \neq h}}^{n} | a_{hj}(s)| \, | x_j(s)| \, | \overline{x_h(s)}| \\
&\geqslant \left[\operatorname{Re}(a_{hh}(s)) - \sum_{\substack{j=1 \\ j \neq h}}^{n} | a_{hj}(s)| \right] | x_h(s)|^2
\end{aligned}
$$

and so by (3.1)

$$
\frac{d}{dt} | x_h |^2 (s) \geqslant 2\delta \, | x_h(s)|^2 > 0. \tag{3.4}
$$

Therefore if $\rho > 0$ is sufficiently small $| x_h(s)| < | x_h(s + \rho)|$ and consequently

$$\| x(s)\| = | x_h(s)| < | x_h(s + \rho)| \leqslant \| x(s + \rho)\|.$$

Now if $t_1 \in (s + \rho, \infty)$ then $\| x(s + \rho)\| < \| x(t_1)\|$; otherwise it would follow from the above that $\| x(t)\|$ would have a relative maximum on the open interval (s, t_1) contradicting Lemma 2.1. Similarly the inequality $\| x(s + \rho)\| < \| x(t_1)\|$ for $t_1 > s + \rho$ implies that if $s + \rho < t_1 < t_2$ then $\| x(t_1)\| < \| x(t_2)\|$; otherwise $\| x(t)\|$ would have a relative maximum on $(s + \rho, t_2)$. This proves that $\| x(t)\|$ is strictly increasing on $(s + \rho, \infty)$ and since $\rho > 0$ can be taken arbitrarily small this proves the lemma.

LEMMA 2.3. *Under the assumptions of Theorem* 1 *if* $x(t)$ *is a solution of* (3.3) *such that* $\| x(t)\|$ *is strictly increasing on* $(-\infty, \infty)$, *then for each* $s \in (-\infty, \infty)$ *there exists an integer* h *such that* $\| x(s)\| = | x_h(s)|$ *and* $\mathrm{Re}(a_{hh}(t)) > 0$ *for all* $t \in (-\infty, \infty)$.

Proof. Assume on the contrary that there exists a number $s \in (-\infty, \infty)$ such that for every integer i between 1 and n with $\| x(s)\| = | x_i(s)|$ there results the inequality $\mathrm{Re}(a_{ii}(t)) < 0$ for all $t \in (-\infty, \infty)$. Partition the integers from 1 to n into two groups I_1 and I_2 such that

$$\begin{aligned} i \in I_1 &\Rightarrow | x_i(s)| < \| x(s)\|, \\ i \in I_2 &\Rightarrow | x_i(s)| = \| x(s)\|. \end{aligned}$$

By continuity there exists a number $\rho_1 > 0$ such that

$$| x_i(t)| < \| x(s)\| \qquad \text{if} \quad i \in I_1 \quad t \in [s, s + \rho_1]. \tag{3.5}$$

If $i \in I_2$ then by assumption $\mathrm{Re}(a_{ii}(t)) < 0$ on $(-\infty, \infty)$ and $\| x(s)\| = | x_1(s)|$. Hence, by (3.1), (3.2) and (3.3)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} | x_i |^2 (s) &= \mathrm{Re}(a_{ii}(s)) \, | x_i(s)|^2 + \mathrm{Re}\left(\sum_{\substack{j=1 \\ j \neq h}}^{n} a_{ij}(s)\, x_j(s)\, \overline{x_i(s)}\right) \\ &\leqslant \mathrm{Re}(a_{ii}(s)) \, | x_i(s)|^2 + \sum_{\substack{j=1 \\ j \neq i}}^{n} | a_{ij}(s)| \, | x_j(s)| \, | \overline{x_i(s)}| \\ &\leqslant \left[\mathrm{Re}(a_{ii}(s)) + \sum_{\substack{j=1 \\ j \neq i}}^{n} | a_{ij}(s)|\right] | x_i(s)|^2 < 0 \end{aligned}$$

so there exists a number $\rho_2 > 0$ such that

$$| x_i(t)| < | x_i(s)| = \| x(s)\| \qquad \text{if} \quad i \in I_2 \,, \quad s \leqslant t \leqslant s + \rho_2 \,. \tag{3.6}$$

Now (3.5) and (3.6) imply that if $\rho = \min\{p_1, p_2)$ then

$$\| x(s+\rho)\| = \max_{1 \leqslant i \leqslant n} | x_i(s+\rho)| < \| x(s)\|$$

and this contradicts the assumption that $\| x(t)\|$ is strictly increasing. This contradiction proves the lemma.

LEMMA 2.4. *Under the assumptions of Theorem 2 if $x(t)$ is a solution of (3.3) such that $\| x(t)\|$ is strictly increasing on $(-\infty, \infty)$ then if $\delta > 0$ is as in (3.1)*

$$\| x(t_1)\| \exp \delta(t_2 - t_1) \leqslant \| x(t_2)\| \qquad \text{if} \quad t_1 \leqslant t_2 . \tag{3.7}$$

Proof. We use the principle of continuous induction. Suppose $x(t)$ is a solution of (3.3) such that $\| x(t)\|$ is strictly increasing and let $s \in (-\infty, \infty)$ be arbitrary. By Lemma 2.3 there exists an integer h such that $\| x(s)\| = | x_h(s)|$ and $\mathrm{Re}(a_{hh}(t)) > 0$ for all $t \in (-\infty, \infty)$. The same computation that gave (3.4) applies here so

$$\frac{1}{| x_h(s)|^2} \frac{d}{dt} | x_h |^2 (s) \geqslant 2\delta.$$

Let α be any number such that $0 < \alpha < \delta$. By continuity there exists a number $\rho(s, \alpha) > 0$ such that $| x_h(t)| \neq 0$ and

$$\frac{1}{| x_h(t)|^2} \frac{d}{dt} | x_h |^2 (t) \geqslant 2\alpha \qquad \text{if} \qquad s \leqslant t \leqslant s + \rho(s, \alpha).$$

Consequently $| x_h(t)|^2 \geqslant | x_h(s)|^2 \exp 2\alpha(t - s)$ if $s \leqslant t \leqslant s + \rho(s, \alpha)$ and since $\| x(s)\| = | x_h(s)|$ and $\| x(t)\| \geqslant | x_h(t)|$,

$$\| x(s)\|^2 \exp 2\alpha(t - s) \leqslant \| x(t)\|^2, \qquad s \leqslant t \leqslant s + \rho(s, \alpha). \tag{3.8}$$

Now let $t_1 \in (-\infty, \infty)$ be arbitrary and define a set of real numbers A such that $\tau \in A$ if and only if $\tau \geqslant t_1$ and $\| x(t_1) \|^2 \exp 2\alpha(t - t_1) \leqslant \| x(t)\|^2$, if $t_1 \leqslant t \leqslant \tau$. A is not empty because $t_1 \in A$. Suppose A were bounded above and let $c = l.u.b.A$. By continuity

$$\| x(t_1)\|^2 \exp 2\alpha(t - t_1) \leqslant \| x(t)\|^2 \qquad \text{if} \quad t_1 \leqslant t \leqslant c.$$

Using (3.8) if $t \in [c, c + \rho(c, \alpha)]$ then

$$\begin{aligned} \| x(t)\|^2 &\geqslant \| x(c)\|^2 \exp 2\alpha(t - c) \geqslant \| x(t_1)\|^2 \exp 2\alpha(t - c) \exp 2\alpha(c - t_1) \\ &= \| x(t_1)\| \exp 2\alpha(t - t_1). \end{aligned}$$

Therefore $c + \rho(c, \alpha) \in A$ and since $c + \rho(c, \alpha) > c$ we have a contradiction. This contradiction shows that A is unbounded and hence

$$\| x(t_1)\|^2 \exp 2\alpha(t_2 - t_1) \leqslant \| x(t_2)\|^2 \qquad \text{if} \quad t_1 \leqslant t_2 .$$

Now α is any number with $0 < \alpha < \delta$ so by continuity the last inequality holds for $\alpha = \delta$ and the inequality (3.7) is proved.

LEMMA 2.5. *Under the assumptions of Theorem* 2 *for each positive integer* m *there exists a* k*-dimensional subspace* V_m *of* K^n *such that if* $x(t)$ *is a nontrivial solution of* (3.3) *with* $x(0) \in V_m$ *then* $\| x(t)\|$ *is strictly increasing on* $[-m, \infty)$.

Proof. Let S be the k-dimensional subspace of K^n defined by

$$S = \{\mathrm{col}(\gamma_1 ,..., \gamma_n) \in K^n \mid \gamma_h = 0 \text{ if } \mathrm{Re}(a_{hh}(t)) < 0 \text{ for } t \in (-\infty, \infty)\}.$$

If $x(t)$ is a nontrivial solution of (3.3) with $x(-m) \in S$ then by Lemma 2.2 $\| x(t)\|$ is strictly increasing on $[-m, \infty)$.

Let $Y(t, s)$ be the $n \times n$ matrix defined for t, $s \in (-\infty, \infty)$ such that $(\partial/\partial t)\, Y = A(t)\, Y$, $Y(s, s) = I$, and define V_m to be the subspace K^n such that

$$V_m = \{c \in K^n \mid c = Y(0, -m)\, a,\ a \in S\}, \qquad \text{for} \quad m = 1, 2,....$$

By the fundamental theory of linear differential equations the matrix $Y(0, -m)$ is nonsingular so dimension V_m = dimension $S = k$.

By the way in which V_m is defined it is obvious that V_m fulfills the assertion of the lemma. This completes the proof of the last of the preliminary lemmas.

Proof of Theorem 2. For each positive integer $m = 1, 2,...$ let V_m be the k-dimensional subspace K^n defined above and $\{c_m{}^j, j = 1,..., k\} \subset V^m$ be an orthonormal basis of V_m, i.e.

$$c_m^{i*} c_m{}^j = 0, \quad i \neq j, \qquad c_m^{i*} c_m{}^i = 1, \quad i, j = 1,..., k. \tag{3.9}$$

By compactness of the unit sphere $\{c \in K^n \mid | c | = 1\}$ in K^n there exists a sequence of integers $\{m_q\}$ and vectors $c^j \in K^n$ such that

$$\lim_{q \to \infty} | c^j - c^j_{m_q} | = 0 \qquad j = 1,..., k. \tag{3.10}$$

Clearly

$$c^{i*} c^j = 0, \quad i \neq j, \qquad c^{i*} c^i = 1, \quad i, j = 1,..., k. \tag{3.11}$$

Let $y^1(t),..., y^k(t)$ be the solutions of (3.3) defined by the initial conditions

$$y^j(0) = c^j, \qquad j = 1,..., k. \tag{3.12}$$

By (3.11) $y^1(t),\ldots, y^k(t)$ are independent since according to (3.11) the values of these solutions at $t = 0$ are independent. Let $x(t)$ be a nontrivial solution of (3.3) of the form

$$x(t) = \alpha_1 y^1(t) + \cdots + \alpha_k y^k(t). \tag{3.13}$$

We will show that $\| x(t)\|$ is strictly increasing on $(-\infty, \infty)$.

Suppose then that t_1 and t_2 are arbitrary real numbers with $t_1 < t_2$. For each integer $q = 1, 2,\ldots$ let $x_q(t)$ be the solution of (3.3) satisfying the initial condition

$$x_q(0) = \alpha_1 c^1_{m_q} + \cdots + \alpha_k c^k_{m_q}. \tag{3.14}$$

Since $x(t)$ is assumed to be a nontrivial solution of (3.3) it follows by (3.9), (3.12) and (3.13) that each $x_q(t)$ is a nontrivial solution of (3.3). Thus since $x_q(0) \in V_{m_q}$ we infer from Lemma 2.5 that $\| x_q(t)\|$ is strictly increasing on the interval $[-m_q, \infty)$ and so

$$\| x_q(t_1)\| < \|x_q(t_2)\| \qquad \text{if} \quad -m_q < t_1 .$$

Now by (3.10), (3.13), (3.14) the equivalence of all norms on the finite dimensional space K^n and continuity of solutions of (3.3) with respect to initial conditions imply that for each $t \in (-\infty, \infty)$ $\| x_q(t)\| \to \| x(t)\|$ as $q \to \infty$. Hence $\| x(t_1)\| \leqslant \| x(t_2)\|$. Since t_1 and t_2 were arbitrary real numbers with $t_1 < t_2$ we have shown that $\| x(t)\|$ is nondecreasing on $(-\infty, \infty)$. If $\| x(t)\|$ were not strictly increasing on $(-\infty, \infty)$ there would exist an open interval on which $\| x(t)\|$ would be constant and this would contradict Lemma 2.1 and the assumption that $x(t)$ is a nontrivial solution of (3.3). This contradiction proves that $\| x(t)\|$ is strictly increasing on $(-\infty, \infty)$ and the proof of the first statement of part (a) of Theorem 2 is complete.

Again let $x(t)$ be a nontrivial solution of (3.3) of the form (3.13). Using what we have just shown and Lemma (2.4) we see that

$$\| x(t_1)\| \exp \delta(t_2 - t_1) \leqslant \| x(t_2)\| \qquad \text{if} \quad t_1 \leqslant t_2 .$$

Therefore $\| x(0)\| \exp \delta t \leqslant \| x(t)\|$ if $t \geqslant 0$ and $\| x(t)\| \leqslant \| x(0)\| \exp \delta t$ if $t \leqslant 0$. Hence $\| x(t)\|$ tends to infinity (zero) as $t \to \infty$ $(-\infty)$ exponentially and the proof of part (a) of Theorem 2 is complete.

We reduce the proof of part (b) of Theorem 2 to part (a) by means of a convenient artifice. Let $B(t) = (b_{ij}(t))$ be the $n \times n$ matrix defined by

$$B(t) = -A(-t).$$

The condition (3.1) implies that $| \operatorname{Re}(b_{ii}(t))| \geqslant \sum_{j=1, j\neq i}^{n} | b_{ij}(t)_i + \delta$ for all $t \in (-\infty, \infty)$. By assumption there are exactly k integers i such that

$\mathrm{Re}(a_{ii}(t)) > 0$ for all $t \in (-\infty, \infty)$ so there are exactly $n - k$ integers i such that $\mathrm{Re}(b_{ii}(t)) > 0$ for all $t \in (-\infty, \infty)$. Hence by part (a) of Theorem 2 applied to the linear differential system

$$\frac{dx}{dt} = B(t)\,x$$

we infer the existence of $n - k$ independent solutions $\hat{y}^1, \ldots, \hat{y}^{n-k}$ of this system such that if $\hat{x}(t)$ is any nontrivial solution of this system of the form $\hat{x}(t) = \alpha_1 \hat{y}^1(t) + \cdots + \alpha_{n-k}\hat{y}^{n-k}$ then $\| \hat{x}(t_1)\| \exp \delta(t_2 - t_1) \leqslant \| \hat{x}(t_2)\|$ if $t_1 \leqslant t_2$. Hence if

$$z^j(t) = \hat{y}^j(-t), \qquad j = 1, \ldots, n - k,$$

then $z^1(t), \ldots, z^{n-k}(t)$ are independent solutions of (3.3) such that if $x(t)$ is any nontrivial solution of (3.3) of the form

$$x(t) = \alpha_1 z^1(t) + \cdots + \alpha_{n-k} z^{n-k}$$

then $\| x(t_2)\| \leqslant \| x(t_1)\| \exp - \delta(t_2 - t_1)$ if $t_1 \leqslant t_2$. This proves part (b) of Theorem 2.

Part (c) of Theorem 2 will follow trivially from parts (a) and (b) provided the n solutions $y^1, \ldots, y^k, z^1, \ldots, z^{n-k}$ are independent. But this follows trivially from the above since otherwise there would exist a nontrivial solution $x(t)$ of (3.3) such that $\| x(t)\|$ is both strictly increasing and strictly decreasing on $(-\infty, \infty)$ which is absurd. Thus the proof is complete.

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